Pre-class Warm-up!!!

True or False?
Let A be a 2 x 2 matrix.

1. If A is not diagonalizable then the characteristic polynomial of A has a repeated root.

a. True ✓ b. False most

2. If the characteristic polynomial of A has a repeated root then A is not diagonalizable.

a. True most b. False ✓

1. We argue by contradiction:
If the char. poly. has no repeated roots then A has 2 distinct e-values so is diagonalizable by 6.2 Thm 3.
Thus A not diagonalizable ⇒ A has a repeated root.

The identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is diagonalizable and its char. poly $(1-\lambda)^2$ has a repeated root.

Section 6.3 Applications involving powers of matrices

We learn:

- a method for finding high powers of matrices using diagonalization
- applications to the stable behavior of populations etc.
- the Cayley-Hamilton theorem

Question. Find A^10 where $A = \begin{bmatrix} -5 & -14 \\ 3 & 8 \end{bmatrix}$

Solution: We diagonalize. Take $P = \begin{bmatrix} 7 & 2 \\ -3 & -1 \end{bmatrix}$

$P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

Steps: $\det(A - \lambda I) = \cdots = (\lambda - 1)(\lambda - 2)$

Solve $(A - \lambda I)v = 0$: $\lambda = 1$, $\begin{bmatrix} -6 & -14 \\ 3 & 7 \end{bmatrix} v = 0$

$\begin{bmatrix} 7 \\ -3 \end{bmatrix}$ is a basis for the 1-eigenspace

$\lambda = 2$, $\begin{bmatrix} -7 & -14 \\ 3 & 6 \end{bmatrix} v = 0$, $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is a basis for the 2-eigenspace so $P = \begin{bmatrix} 7 & 2 \\ -3 & -1 \end{bmatrix}$ works.

Observe $\overbrace{\underbrace{P^{-1}AP}\,\underbrace{P^{-1}AP}\,\underbrace{P^{-1}AP} \cdots P^{-1}AP}^{10 \text{ times}} = P^{-1}A^{10}P$

$= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^{10} = \begin{bmatrix} 1 & 0 \\ 0 & 2^{10} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1024 \end{bmatrix}$

Finally $A^{10} = P(P^{-1}A^{10}P)P^{-1}$

Find $P^{-1} = \frac{1}{-1} \begin{bmatrix} -1 & -2 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & -7 \end{bmatrix}$

$A^{10} = \begin{bmatrix} 7 & 2 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1024 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -3 & -7 \end{bmatrix}$

$= \begin{bmatrix} 7 & 2 \cdot 1024 \\ -3 & -1024 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -3 & -7 \end{bmatrix}$

$= \begin{bmatrix} 7 - 6 \cdot 1024 & 14 - 14 \cdot 1024 \\ -3 + 3 \cdot 1024 & -6 + 7 \cdot 1024 \end{bmatrix}$

Question. Find A^10 where $A = \begin{bmatrix} -5 & -14 \\ 3 & 8 \end{bmatrix}$

Solution: We diagonalize. Take $P = \begin{bmatrix} 7 & 2 \\ -3 & -1 \end{bmatrix}$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Steps: $\det(A - \lambda I) = \cdots = (\lambda - 1)(\lambda - 2)$

Solve $(A - \lambda I)v = 0$: $\lambda = 1$, $\begin{bmatrix} -6 & -14 \\ 3 & 7 \end{bmatrix} v = 0$

$\begin{bmatrix} 7 \\ -3 \end{bmatrix}$ is a basis for the 1-eigenspace

$\lambda = 2$, $\begin{bmatrix} -7 & -14 \\ 3 & 6 \end{bmatrix} v = 0$, $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is a basis for the 2-eigenspace so $P = \begin{bmatrix} 7 & 2 \\ -3 & -1 \end{bmatrix}$ works.

Note that

$$(P^{-1}AP)^{10} = \overbrace{P^{-1}A\cancel{P}\,\cancel{P^{-1}}A\cancel{P}\,\cancel{P^{-1}}A\cancel{P}\cdots\cancel{P^{-1}}AP}^{10 \text{ times}}$$

$$= P^{-1}A^{10}P$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^{10} = \begin{bmatrix} 1 & 0 \\ 0 & 1024 \end{bmatrix}$$

Finally

$$A^{10} = P(P^{-1}A^{10}P)P^{-1} = P\begin{bmatrix} 1 & 0 \\ 0 & 1024 \end{bmatrix}P^{-1}$$

$$= \begin{bmatrix} 7 & 2 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1024 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 2 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 \cdot 1024 & -7 \cdot 1024 \end{bmatrix}$$

$$= \begin{bmatrix} 7 - 6 \cdot 1024 & 14 - 14 \cdot 1024 \\ -3 + 3 \cdot 1024 & -6 + 7 \cdot 1024 \end{bmatrix}$$

Question:
At a debate between candidates A and B, of the people who started supporting A, 0.8 stay with A, 0.2 change to B.
Of the people who started supporting B, 0.9 stay with B, 0.1 change to A. After a large number of debates, what proportion of people support each candidate?

Another question:
Each year, of the people who live in the center of a city, 0.8 of them stay in the center and 0.2 of them move to the suburbs.
Of the people who live in the suburbs, 0.9 of them stay in the suburbs and 0.1 of them move to the center. After many years, what proportion of people live in the center and what proportion in the suburbs?

Yet another question:
After k years the number of rabbits in a region is r_k and the number of foxes is f_k, and these satisfy

$$r_k = 0.8 r_{k-1} + 0.1 f_{k-1}$$
$$f_k = 0.2 r_{k-1} + 0.9 f_{k-1}$$

After many years, what is the proportion of rabbits to foxes?

Question:
Which of these three questions do you think is the easiest to solve?

a. the first
b. the second
c. the third
d. None of them

Question:

At a debate between candidates A and B, of the people who started supporting A, 0.8 stay with A, 0.2 change to B.
Of the people who started supporting B, 0.9 stay with B, 0.1 change to A. After a large number of debates, what proportion of people support each candidate?

Solution. Let S_k^A, S_k^B be the number of supporters of A and B after debate k.

Then $S_k^A = 0.8 S_{k-1}^A + 0.1 S_{k-1}^B$

$S_k^B = 0.2 S_{k-1}^A + 0.9 S_{k-1}^B$

$$\begin{bmatrix} S_k^A \\ S_k^B \end{bmatrix} = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \begin{bmatrix} S_{k-1}^A \\ S_{k-1}^B \end{bmatrix}$$

$$= \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}^k \begin{bmatrix} S_0^A \\ S_0^B \end{bmatrix}$$

Thus about $\frac{1}{3}$ support A, $\frac{2}{3}$ support B eventually

Diagonalize $\begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$.

Char. poly. $= \lambda^2 - 1.7\lambda + 0.7 = \frac{1}{10}(10\lambda^2 - 17\lambda + 7)$

$= \frac{1}{10}(10\lambda - 7)(\lambda - 1)$

e-values: $\lambda = 0.7, 1$

$\lambda = 1$: Solve $\begin{bmatrix} -0.2 & 0.1 \\ 0.2 & -0.1 \end{bmatrix} v = 0$, $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda = 0.7$ Solve $\begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0.2 \end{bmatrix} v = 0$ $v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$ $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 0.7 \end{bmatrix}$

$P^{-1} = \frac{1}{-3}\begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$

$$\begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}^k = P(P^{-1}AP)^k P^{-1} \simeq P\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} P^{-1}$$ when k is large.

$$= \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

$$\begin{matrix} S_k^A \\ S_k^B \end{matrix} \approx \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix} \begin{bmatrix} S_0^A \\ S_0^B \end{bmatrix} = \begin{bmatrix} \frac{1}{3}(S_0^A + S_0^B) \\ \frac{2}{3}(S_0^A + S_0^B) \end{bmatrix}$$

The Cayley-Hamilton theorem.
Let p_A be the characteristic polynomial of a square matrix A, so

$$p_A = \det(A - \lambda I)$$

Then p_A(A) = 0 (the zero matrix).

Example: $A = \begin{bmatrix} -5 & -14 \\ 3 & 8 \end{bmatrix}$, $p_A(\lambda) = \lambda^2 - 3\lambda + 2$

$$A^2 = \begin{bmatrix} -17 & -42 \\ 9 & 22 \end{bmatrix}$$

The CH theorem says $A^2 - 3A + 2I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} -17 & -42 \\ 9 & 22 \end{bmatrix} - \begin{bmatrix} -15 & -42 \\ 9 & 24 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Like 6.3 questions 15-24 (not on the HW):
Use the CH theorem to find A^{-1}, A^3, A^4.

Solution: Start with $p_A(A) = A^2 - 3A + 2I$

Multiply by A^{-1}: $A - 3I + 2A^{-1} = 0$

$$A^{-1} = \frac{1}{2}(3I - A)$$

$$= \frac{1}{2}\left(\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} -5 & -14 \\ 3 & 8 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 8 & 14 \\ -3 & -5 \end{bmatrix}$$

Multiply by A: $A^3 - 3A^2 + 2A = 0$

$$A^3 = 3A^2 - 2A = \cdots$$

The Cayley-Hamilton theorem.
Let p_A be the characteristic polynomial of a square matrix A.
Then p_A(A) = 0 (the zero matrix).

Proof: We use the adjoint matrix from page 199.

$$\operatorname{adj}(B) = [B_{ij}]^T$$

$$B_{ij} = (-1)^{i+j} \det(\,ij\text{-minor of } B\,)$$

$B \cdot \operatorname{adj}(B) = |B| \cdot I$ for every matrix B, so

$$(A-\lambda I)\operatorname{adj}(A-\lambda I) = \det(A-\lambda I)\cdot I = p_A\, I$$

Write

$$\operatorname{adj}(A-\lambda I) = Q_0 + Q_1\lambda + \cdots + Q_{n-1}\lambda^{n-1}$$

$$p_A = c_0 + c_1\lambda + \cdots + c_n\lambda^n$$

Now $(A-\lambda I)\operatorname{adj}(A-\lambda I)$

$$= AQ_0 - Q_0\lambda + AQ_1\lambda - Q_1\lambda^2 + \cdots + AQ_{n-1}\lambda^{n-1}$$

$$= c_0 I + c_1\lambda I + \cdots + c_n\lambda^n I$$

Equate powers of λ to get:

$$\left.\begin{array}{lcll} AQ_0 & = & c_0 I & \\ AQ_1 - Q_0 & = & c_1 I & \text{multiply by } A \\ \vdots & & & A^2 \\ AQ_{n-1} - Q_{n-2} & = & c_{n-1} I & A^{n-1} \\ -Q_{n-1} & = & c_n I & A^n \end{array}\right] \text{add}$$

$$AQ_0 + A^2Q_1 - AQ_0 + \cdots + A^nQ_{n-1} - A^{n-1}Q_{n-2} - A^nQ_{n-1}$$

$$= 0$$

$$= c_0 I + c_1 A + c_2 A^2 + \cdots + c_n A^n$$

$$= p_A(A).$$